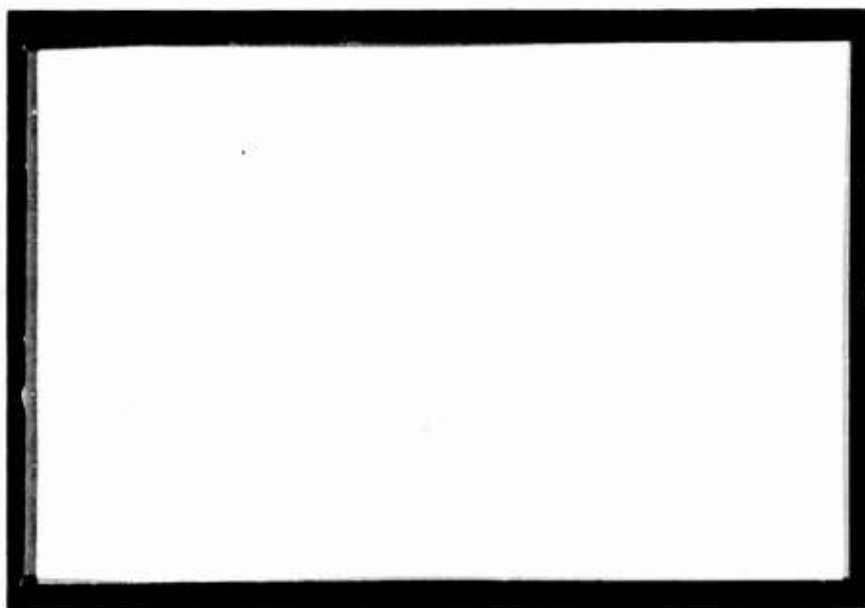
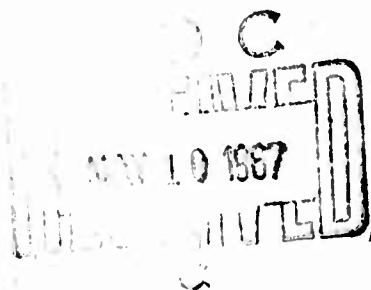


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ON THE DETERMINATION OF ORBITS BY THE
SOLUTION OF A SYSTEM OF
INTEGRAL EQUATIONS

V. T. Gontkovskaja

Translated from Russian by L. B. Rall

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ABSTRACT

The determination of an orbit by using two heliocentric positions of a celestial body is a boundary-value problem of mathematical physics, and may be reduced to the solution of a nonlinear integral equation. Methods of successive approximation to the solution are discussed, and convergence conditions and error estimates are derived.

The problems of perturbed motion and the determination of orbits based on three geocentric observations are also considered.

ON THE DETERMINATION OF ORBITS BY THE SOLUTION
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One of the basic problems of celestial mechanics - the determination of an orbit from two heliocentric positions - may be considered to be a boundary-value problem of mathematical physics. Until recently, such problems have been solved by integrating differential equations with given initial conditions. This reduction from one problem to the other turns out to be possible only because the perturbed motion differs only slightly from Keplerian motion in the two-body problem, the general solution of which is known for any statement of it (either as an initial-value or a boundary-value problem).

At the present time, a whole series of problems has arisen in connection with the conquest of space in which the perturbations are very large, and the Keplerian orbit cannot serve as an approximate solution. Thus, the determination of an orbit in such cases is only possible directly from the boundary-value problem, for example, by reducing it to an integral equation.

This method was first proposed by Bucerius in his fundamental paper, "Bahnbestimmung als Randwertproblem." Bucerius not only stated the problem, but also proposed a method for its solution. However, the method of successive

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approximation proposed by him for the solution of the system of integral equations, to which the problem of determination of orbits from two heliocentric positions reduces, contains a whole series of inaccuracies and does not have a rigorous foundation. Indeed, for the method of successive approximation proposed by Bucerius for the solution of systems of integral equations differing somewhat from those given here, some conditions given for convergence are obtained as a result of non-rigorous reasoning, and, what is very essential, Bucerius does not have any error estimates whatever for his method.

In this paper, two methods of successive approximation are considered which permit the solution of the problem posed by Bucerius. Sufficient conditions are obtained for the convergence of the sequence of approximations to the solution, the uniqueness of the solution obtained is proved, and, in addition, estimates are given for the error committed by terminating the calculation at the n th approximation.

§1. Statement of the Problem

The differential equations of motion in vector form are:

$$\ddot{\underline{r}}(t) = -k^2(1+m) \frac{\underline{r}(t)}{|\underline{r}(t)|^3}. \quad (1)$$

We introduce the vector $\underline{r}_0(t)$, which satisfies the equation of uniform rectilinear motion

$$\ddot{\underline{r}}_0(t) = 0,$$

in the form

$$\underline{r}_0(t) = \frac{t_2 - t}{t_2 - t_1} \underline{r}_1 + \frac{t - t_1}{t_2 - t_1} \underline{r}_2 ,$$

where \underline{r}_1 and \underline{r}_2 are determined by two heliocentric positions at the times t_1 and t_2 . Thus, the vector $\underline{r}(t) - \underline{r}_0(t)$ satisfies a second-order differential equation with zero boundary conditions, which properly completely define a solution.

By means of the Green's function

$$G(t, t') = \frac{(t - t_1)(t_2 - t')}{t_2 - t_1} , \quad t \leq t' ,$$

$$G(t, t') = G(t', t)$$

the problem may be reduced to the solution of the nonlinear integral equation

$$\underline{r}(t) = \underline{r}_0(t) + k^2(1+m) \int_{t_1}^{t_2} G(t, t') \frac{\underline{r}(t')}{|\underline{r}(t')|^3} dt' . \quad (2)$$

If the time t_1 is taken to be the beginning of the measurement of time, and the difference $t_2 - t_1$ as a unit of time, then the time t varies over the interval $[0, 1]$. The kernel of the integral equation takes the form

$$K(t, t') = \begin{cases} t(1 - t'), & t \leq t' \\ t'(1 - t), & t \geq t' \end{cases}$$

and equation (2) may be written in the following fashion

$$\underline{r}(t) = \underline{r}_0(t) + \tau^2 \int_0^1 K(t, t') \frac{\underline{r}(t')}{|\underline{r}(t')|^3} dt' , \quad (3)$$

where

$$\tau = k\sqrt{1+m} (t_2 - t_1) .$$

Since the vectors $\underline{r}, \underline{r}_1, \underline{r}_2$ are coplanar, the relationship

$$\underline{r}(t) = n_1(t) \underline{r}_1 + n_2(t) \underline{r}_2 , \quad (4)$$

holds, where

$$n_1(t) = n_1(t, t_1, t_2, \underline{r}_1, \underline{r}_2) ,$$

$$n_2(t) = n_2(t, t_1, t_2, \underline{r}_1, \underline{r}_2) .$$

Interpreted geometrically, n_1, n_2 are the ratios of the areas of the triangles

$$n_1(t) = \frac{[\underline{r} \underline{r}_2]}{[\underline{r}_1 \underline{r}_2]} , \quad n_2(t) = \frac{[\underline{r}_1 \underline{r}]}{[\underline{r}_1 \underline{r}_2]} .$$

With the help of relationship (4), the integral equation (4) reduced to the two integral equations

$$\left. \begin{aligned} n_1(t) &= 1 - t - \tau^2 \int_0^1 K(t, t') \frac{n_1(t')}{|n_1(t') \underline{r}_1 + n_2(t') \underline{r}_2|^3} dt' , \\ n_2(t) &= t + \tau^2 \int_0^1 K(t, t') \frac{n_2(t')}{|n_1(t') \underline{r}_1 + n_2(t') \underline{r}_2|^3} dt' \end{aligned} \right\} (5)$$

$$|n_1(t') \underline{r}_1 + n_2(t') \underline{r}_2|^3 = |n_1^2(t') r_1^2 + n_2^2(t') r_2^2 + 2n_1(t') n_2(t') (\underline{r}_1 \underline{r}_2)|^{3/2} .$$

§2. A Method of Successive Approximations Applied to the Nonlinear Integral Equation

To solve equation (3), Bucerius (1950) proposed an iterative method which consisted of a sequence of applications of the mean-value theorem to the integral on the right side.

Suppose that $\underline{r}_0(t)$ is taken as the initial approximation to the solution of the integral equation (and the initial approximations to the ratios of the area as $\frac{\tau_1}{\tau}, \frac{\tau_2}{\tau}$), then the first approximation is

$$\underline{r}_1(t) = \underline{r}_0(t) + \tau^2 \int_0^1 K(t, t') \frac{\underline{r}_0(t')}{|\underline{r}_0(t')|^3} dt' = \underline{r}_0(t) + \frac{\tau^2}{r_1^3} \int_0^1 K(t, t') \underline{r}_0(t') dt'.$$

The average value of the quantity $|\underline{r}_0(t')|^3$ in the interval $0 < t' < 1$ is denoted by r_1^3 .

Introducing the notation

$$\lambda_1 = \frac{\tau^2}{r_1^3},$$

then

$$\underline{r}_1(t) = \underline{r}_0(t) + \lambda_1 \int_0^1 K(t, t') \underline{r}_0(t') dt'.$$

The second approximation has the form

$$\underline{r}_2(t) = \underline{r}_0(t) + \tau^2 \int_0^1 K(t, t') \frac{\underline{r}_1(t')}{|\underline{r}_1(t')|^3} dt'.$$

If the expression for $\underline{r}_1(t')$ is substituted into it, then

$$\underline{r}_2(t) = \underline{r}_0(t) + \lambda_2 \int_0^1 K(t, t') \underline{r}_0(t') dt' + \lambda_2 \lambda_1 \int_0^1 K_2(t, t') \underline{r}_0(t') dt',$$

where

$$\lambda_2 = \frac{\tau^2}{r_2^3},$$

and $K_2(t, t')$ is the second iterated kernel.

For the v th iteration, one will obtain

$$\begin{aligned} \underline{r}_\nu(t) = & \underline{r}_0(t) + \lambda_\nu \int_0^1 K(t, t') \underline{r}_0(t') dt' + \lambda_\nu \lambda_{\nu-1} \int_0^1 K_2(t, t') \underline{r}_0(t') dt' + \dots + \\ & + \lambda_\nu \lambda_{\nu-1} \dots \lambda_1 \int_0^1 K_\nu(t, t') \underline{r}_0(t') dt' , \end{aligned} \quad (6)$$

$$\lambda_\nu = \frac{\tau^2}{r_\nu^3} ,$$

$K_\nu(t, t')$ is the ν th iterated kernel.

All λ_ν satisfy the inequality $\lambda_\nu \leq \lambda = \frac{\tau^2}{h^3}$, where $h = \min \underline{r}(t)$ ($0 \leq t \leq 1$) is the perpendicular issuing from the origin of coordinates to the chord $\underline{r}_1 \underline{r}_2$. Letting ν go to infinity and replacing all λ_ν in (6) by $\frac{\tau^2}{h^3}$, one may consider the series obtained to be the Neumann series for the equation

$$\underline{r}(t) = \underline{r}_0(t) + \frac{\tau^2}{h^3} \int_0^1 K(t, t') \underline{r}(t') dt' .$$

A lower bound for the radius of convergence of this series is (Wiarda, 1930)

$$\lambda \leq \left(\int_0^1 \int_0^1 [K(t, t')]^2 dt dt' \right)^{-1/2} = 3\sqrt{10} ,$$

or $\lambda \leq \lambda_1$, where λ_1 is the first characteristic value of the kernel $K(t, t')$.

Consequently, if all $\lambda_\nu \leq \lambda_1$, then the series (6) converges.

In applying the iterative process described above to the solution of equation (3), Bucerius, generally speaking, started from the mistaken assumption that the λ_ν do not depend on time, in the sense that they are the same for each orbit. Indeed, $\lambda_\nu = \lambda_\nu(t)$, and one has to prove the assertion that all $r_\nu \geq h$, from which the convergence of the process of successive approximation follows.

§3. Solution of the Nonlinear Integral Equation by the Method of Successive Approximations.

We apply to equation (3) the method of successive approximations in the following form.

We first take $\tilde{r}(t) = \tilde{r}_0(t)$ and define $\tilde{r}_1(t)$ by the relationship

$$\tilde{r}_1(t) = \tilde{r}_0(t) + \tau^2 \int_0^1 K(t, t') \frac{\tilde{r}_0(t')}{r_0^3(t')} dt'.$$

Continuing in the same fashion, we obtain an infinite sequence of functions

$$\tilde{r}_0(t), \tilde{r}_1(t), \dots, \tilde{r}_n(t), \dots,$$

which satisfy the recurrence relations

$$\tilde{r}_n(t) = \tilde{r}_0(t) + \tau^2 \int_0^1 K(t, t') \frac{\tilde{r}_{n-1}(t')}{r_{n-1}^3(t')} dt'. \quad (7)$$

We shall show that the sequence of vectors $\{\tilde{r}_n(t)\}_{n=0}^{\infty}$ converges to a function which is a solution of equation (3).

For the proof, consider the difference of two successive approximations

$$\tilde{r}_n(t) - \tilde{r}_{n-1}(t) = \tau^2 \int_0^1 K(t, t') \left[\frac{\tilde{r}_{n-1}(t')}{r_{n-1}^3(t')} - \frac{\tilde{r}_{n-2}(t')}{r_{n-2}^3(t')} \right] dt'.$$

The difference enclosed in square brackets may be written in the following manner,

$$\begin{aligned} & \left[\frac{\tilde{r}_{n-1}(t')}{r_{n-1}^3(t')} - \frac{\tilde{r}_{n-2}(t')}{r_{n-1}^3(t')} \right] + \left[\frac{\tilde{r}_{n-2}(t')}{r_{n-1}^3(t')} - \frac{\tilde{r}_{n-2}(t')}{r_{n-2}^3(t')} \right] = \frac{\tilde{r}_{n-1}(t') - \tilde{r}_{n-2}(t')}{r_{n-1}^3(t')} + \\ & + \tilde{r}_{n-2}(t') \frac{r_{n-2}(t') - r_{n-1}(t')}{r_{n-1}(t') r_{n-2}(t')} \left[\frac{1}{r_{n-1}^2(t')} + \frac{1}{r_{n-1}(t') r_{n-2}(t')} + \frac{1}{r_{n-2}^2(t')} \right]. \end{aligned}$$

In so far as

$$\int_0^1 K(t, t') dt' \leq \frac{1}{8} \quad \text{for } 0 \leq t \leq 1,$$

$$|\tilde{r}_n(t) - \tilde{r}_{n-1}(t)| \leq \frac{\tau^2}{8} \left\{ \frac{|\tilde{r}_{n-1} - \tilde{r}_{n-2}|}{r_{n-1}^3} + \left| \tilde{r}_{n-2} \frac{r_{n-2} - r_{n-1}}{r_{n-1} r_{n-2}} \left(\frac{1}{r_{n-1}^2} + \frac{1}{r_{n-1} r_{n-2}} + \frac{1}{r_{n-2}^2} \right) \right| \right\},$$

and since the difference of two sides of a triangle is less than the third and

$\left| \frac{\tilde{r}_{n-2}}{r_{n-2}} \right| \leq 1$, one has that

$$|\tilde{r}_n(t) - \tilde{r}_{n-1}(t)| \leq \frac{\tau^2}{2} \frac{\max_{0 \leq t \leq 1} |\tilde{r}_{n-1}(t) - \tilde{r}_{n-2}(t)|}{\min_{0 \leq t \leq 1} \{r_{n-1}^3, r_{n-2}^3\}}, \quad (8)$$

or

$$|\tilde{r}_n(t) - \tilde{r}_{n-1}(t)| \leq \max_{0 \leq t \leq 1} |\tilde{r}_1(t) - \tilde{r}_0(t)| \left(\frac{\tau^2}{2 \min_{0 \leq t \leq 1} r_k^3} \right)^{n-1}, \quad (9)$$

$$k = 0, 1, 2, \dots, n-1.$$

Setting

$$\tilde{r}_n(t) - \tilde{r}_{n-1}(t) = \Delta_{n-1}(t),$$

we note that

$$\tilde{r}_n(t) = \tilde{r}_0(t) + [\tilde{r}_1(t) - \tilde{r}_0(t)] + [\tilde{r}_2(t) - \tilde{r}_1(t)] + \dots + [\tilde{r}_n(t) - \tilde{r}_{n-1}(t)] = \tilde{r}_0(t) + \sum_{k=0}^{n-1} \Delta_k(t)$$

and

$$r_n(t) \leq r_0(t) + \Delta_0(t) \left[1 + \frac{\tau^2}{2 \min_{0 \leq t \leq 1} r_n^3(t)} + \dots + \left(\frac{\tau^2}{2 \min_{0 \leq t \leq 1} r_n^3(t)} \right)^{n-1} \right].$$

The geometric progression enclosed in square brackets converges for

$$\frac{\tau^2}{2 \min_{0 \leq t \leq 1} r_n^3(t)} < 1. \quad (10)$$

Obviously, if condition (10) is satisfied, the sequence $\{r_n(t)\}$ converges to the function

$$\underline{R}(t) = \lim_{n \rightarrow \infty} \{r_n(t)\}$$

by virtue of inequality (9), uniformly, moreover, on the interval $0 \leq t \leq 1$ in as much as the estimate (9) is independent of t .

Now,

$$\frac{\tau^2}{2 \min_{0 \leq t \leq 1} R^3(t)} < 1, \quad (11)$$

since (10) is valid for all $r_n(t)$, the limit of which is $\underline{R}(t)$.

The function $\underline{R}(t)$ satisfies equation (3). In fact, by virtue of the uniform convergence of the sequence $\{r_n(t)\}$ to $\underline{R}(t)$, by letting $n \rightarrow \infty$ in formula (7), one obtains that

$$\underline{R}(t) = r_0(t) + \tau^2 \int_0^1 K(t, t') \frac{\underline{R}(t')}{R^3(t')} dt'.$$

We shall prove that $\underline{R}(t)$ is the unique solution of equation (3) in the region

$$\frac{\tau^2}{2 \min_{0 \leq t \leq 1} r^3(t)} < 1. \quad (12)$$

Indeed, suppose that there exists another solution $\rho(t)$ which satisfies condition (12). Then, by performing a calculation similar to the preceding, one obtains that

$$\left| \frac{\tilde{R}(t')}{R^3(t')} - \frac{\rho(t')}{\rho^3(t')} \right| \leq \frac{4|\rho(t') - \tilde{R}(t')|}{\min_{0 \leq t' \leq 1} [R^3(t'), \rho^3(t')]}.$$

Consequently,

$$|\tilde{R}(t) - \rho(t)| = \left| \tau^2 \int_0^1 K(t, t') \left[\frac{\tilde{R}(t')}{R^3(t')} - \frac{\rho(t')}{\rho^3(t')} \right] dt' \right| \leq \frac{\tau^2}{2} \frac{|\tilde{R}(t) - \rho(t)|}{\min_{0 \leq t \leq 1} [R^3(t), \rho^3(t)]}, \quad (0 \leq t \leq 1),$$

or

$$|\tilde{R}(t) - \rho(t)| \left(1 - \frac{1}{2} \frac{\tau^2}{\min_{0 \leq t \leq 1} [R^3(t), \rho^3(t)]} \right) \leq 0.$$

The quantity in parentheses is positive, by virtue of the satisfaction of conditions (11) and (12). Dividing both sides of the above inequality by it, one obtains that

$$|\tilde{R}(t) - \rho(t)| \leq 0, \quad (0 \leq t \leq 1),$$

from which it follows that

$$|\tilde{R}(t) - \rho(t)| \equiv 0, \quad (0 \leq t \leq 1).$$

We now derive a condition which guarantees the satisfaction of inequality (10).

It follows from equation (7) that

$$r_n(t) \geq r_0(t) - \frac{\tau^2}{8 \min_{0 \leq t \leq 1} r_{n-1}^2(t)}. \quad (13)$$

It is easy to see that $r_0(t)$ is larger than the altitude h from the origin of coordinates to the chord joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) .

Then,

$$\min_{0 \leq t \leq 1} r_n(t) \geq h - \frac{\tau^2}{8 \min_{0 \leq t \leq 1} r_{n-1}^2(t)}. \quad (14)$$

Assume that

$$\min_{0 \leq t \leq 1} r_m(t) \geq kh, \quad (m = 0, 1, \dots, n-1), \quad (15)$$

where $k < 1$ is a positive number which we shall define below. Note that this inequality is valid for $m = 0$. We assume that it is valid for all m to $n-1$, inclusive. With the help of inequalities (14) and (15), we obtain that

$$\min_{0 \leq t \leq 1} r_n(t) \geq h - \frac{\tau^2}{8k^2 h^2}.$$

Requiring that

$$h - \frac{\tau^2}{8k^2 h^2} \geq kh, \quad (16)$$

we obtain that $\min_{0 \leq t \leq 1} r_n(t) \geq kh$, so that we have proved this inequality for all n by means of the principle of mathematical induction. Thus, by virtue of (10), the inequality

$$\frac{\tau^2}{2k^3 h^3} < 1, \quad (17)$$

is a sufficient condition for convergence.

Both inequalities (16) and (17) are satisfied if one chooses $k \leq \frac{4}{5}$.

Actually, (16) is a consequence of inequality (17) if one requires further that

$$h - \frac{2k^3 h^3}{8k^2 h^2} \geq kh ,$$

from which

$$1 - \frac{1}{4}k \geq k ,$$

or

$$k \leq \frac{4}{5} .$$

Consequently, the process of successive approximations described above converges to a solution of our integral equation in case that

$$h^3 \geq \frac{125}{128} \tau^2 ,$$

for which

$$\min_{0 \leq t \leq 1} r_n(t) \geq \frac{4}{5} h .$$

We now obtain an estimate for the error committed if the calculation is halted at the n th approximation:

$$|r(t) - r_n(t)| \leq \sum_{k=n}^{\infty} \Delta_k = \frac{\alpha^n}{1-\alpha} \Delta_0 ,$$

where

$$\alpha = \frac{\tau^2}{2 \min_{0 \leq t \leq 1} r_n^3(t)} \leq \frac{125}{128} \frac{\tau^2}{h^3} .$$

Since

$$\Delta_0 = \tau^2 \left| \int_0^1 K(t, t') \frac{r_0(t')}{r_0^3(t')} dt' \right| \leq \frac{\tau^2}{8 \min_{0 \leq t \leq 1} r_0^2(t)} \leq \frac{1}{8} \frac{\tau^2}{h^2} \leq \frac{16}{125} h ,$$

finally,

$$|r(t) - r_n(t)| \leq \frac{16}{125} h \frac{\left(\frac{125}{128} \frac{\tau^2}{h^3} \right)^n}{1 - \frac{125}{128} \frac{\tau^2}{h^3}} .$$

§4. The Application of Newton's Method to the Solution of a System of Nonlinear Integral Equations.

Newton's method, sometimes called the method of tangents, is one of the most effective methods for obtaining solutions of algebraic equations in the case that an initial approximation is known for the value of the solution. The successive approximations are defined by a formula of the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} .$$

This method extends to systems of algebraic equations. It may be applied (Kantorovič, 1949) to classes of arbitrary nonlinear equations, in particular, to classes of nonlinear integral equations.

We shall apply Newton's method to the system of nonlinear integral equations

$$\left. \begin{aligned} \underline{n}_1(t) &= \underline{n}_1^0(t) - 1 - \tau^2 \int_0^1 U_1(t, t', \underline{n}_1(t'), \underline{n}_2(t')) dt' , \\ \underline{n}_2(t) &= \underline{n}_2^0(t) - 1 - \tau^2 \int_0^1 U_2(t, t', \underline{n}_1(t'), \underline{n}_2(t')) dt' , \end{aligned} \right\} \quad (18)$$

for which initial approximations to the solutions may be taken to be $\underline{n}_1^0(t), \underline{n}_2^0(t)$.

To obtain the first approximations, one adds to $\underline{n}_1^0(t), \underline{n}_2^0(t)$ the corrections

$$\begin{aligned} \Delta_2^1 &= \underline{n}_1^1(t) - \underline{n}_1^0(t) , \\ \Delta_2^1 &= \underline{n}_2^1(t) - \underline{n}_2^0(t) , \end{aligned}$$

determined from the linear integral equations

$$\begin{aligned} \Delta_1^1 - \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_1}{\partial n_1} \right)_0 \Delta_1^1 + \left(\frac{\partial U_1}{\partial n_2} \right)_0 \Delta_2^1 \right\} dt' &= \tau^2 \int_0^1 U_1(t, t', \underline{n}_1^0(t'), \underline{n}_2^0(t')) dt' , \\ \Delta_2^1 - \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_2}{\partial n_1} \right)_0 \Delta_1^1 + \left(\frac{\partial U_2}{\partial n_2} \right)_0 \Delta_2^1 \right\} dt' &= \tau^2 \int_0^1 U_2(t, t', \underline{n}_1^0(t'), \underline{n}_2^0(t')) dt' , \end{aligned} \quad (19)$$

The second approximations $\underline{n}_1^2(t)$, $\underline{n}_2^2(t)$ will be obtained by determining Δ_1^2 , Δ_2^2 from the system

$$\begin{aligned} \Delta_1^2 - \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_1}{\partial n_1} \right)_1 \Delta_1^2 + \left(\frac{\partial U_1}{\partial n_2} \right)_1 \Delta_2^2 \right\} dt' &= \underline{n}_1^0(t) - \underline{n}_1^1(t) + \tau^2 \int_0^1 U_1(t, t', \underline{n}_1^1(t'), \underline{n}_2^1(t')) dt' , \\ \Delta_2^2 - \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_2}{\partial n_1} \right)_1 \Delta_1^2 + \left(\frac{\partial U_2}{\partial n_2} \right)_1 \Delta_2^2 \right\} dt' &= \underline{n}_2^0(t) - \underline{n}_2^1(t) + \tau^2 \int_0^1 U_2(t, t', \underline{n}_1^1(t'), \underline{n}_2^1(t')) dt' \end{aligned}$$

with

$$\begin{aligned} \underline{n}_1^2(t) &= \underline{n}_1^1(t) + \Delta_1^2 , \\ \underline{n}_2^2(t) &= \underline{n}_2^1(t) + \Delta_2^2 , \end{aligned}$$

finally, the n th approximation is determined by solving the following system of linear integral equations:

$$\left. \begin{aligned} \Delta_1^n - \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_1}{\partial n_1} \right)_{n-1} \Delta_1^n + \left(\frac{\partial U_1}{\partial n_2} \right)_{n-1} \Delta_2^n \right\} dt' &= \\ &= \underline{n}_1^0(t) - \underline{n}_1^{n-1}(t) + \tau^2 \int_0^1 U_1(t, t', \underline{n}_1^{n-1}(t'), \underline{n}_2^{n-1}(t')) dt' , \\ \Delta_2^n - \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_2}{\partial n_1} \right)_{n-1} \Delta_1^n + \left(\frac{\partial U_2}{\partial n_2} \right)_{n-1} \Delta_2^n \right\} dt' &= \\ &= \underline{n}_2^0(t) - \underline{n}_2^{n-1}(t) + \tau^2 \int_0^1 U_2(t, t', \underline{n}_1^{n-1}(t'), \underline{n}_2^{n-1}(t')) dt' . \end{aligned} \right\} \quad (20)$$

In the case of interest to us, system (5), the functions U_i and the derivatives $\frac{\partial U_1}{\partial n_1}, \frac{\partial U_2}{\partial n_1}$ ($i = 1, 2$) have the following form

$$\left. \begin{aligned} U_1 &= \frac{K(t, t') n_1(t')}{[n_1^2 r_1^2 + n_2^2 + 2n_1 n_2 (r_1 r_2)]^{3/2}}, \\ U_2 &= \frac{K(t, t') n_2(t')}{[n_1^2 r_1^2 + n_2^2 r_2^2 + 2n_1 n_2 (r_1 r_2)]^{3/2}}, \\ \frac{\partial U_1}{\partial n_2} &= -\frac{3K(t, t') n_1(t') [n_2 r_2^2 + n_1 (r_1 r_2)]}{r^5}, \\ \frac{\partial U_1}{\partial n_1} &= \frac{K(t, t')}{r^3} \left\{ 1 - \frac{3n_1 [n_1 r_1^2 + n_2 (r_1 r_2)]}{r^2} \right\}, \\ \frac{\partial U_2}{\partial n_1} &= \frac{3K(t, t') n_2(t') [n_1 r_1^2 + n_2 (r_1 r_2)]}{r^5}, \\ \frac{\partial U_2}{\partial n_2} &= \frac{K(t, t')}{r^3} \left\{ 1 - \frac{3n_2 [n_2 r_2^2 + n_1 (r_1 r_2)]}{r^2} \right\}, \end{aligned} \right\} \quad (21)$$

moreover,

$$\left. \begin{aligned} \left| \int_0^1 U_i(t, t') dt' \right| &\leq \frac{1}{8} \frac{\max_{0 \leq t \leq 1} \{n_1(t), n_2(t)\}}{\min_{0 \leq t \leq 1} r^3(t)}, \\ \left| \int_0^1 \frac{\partial U_i}{\partial n_1} dt' \right| &\leq \frac{1}{8} \frac{M^2 \max_{0 \leq t \leq 1} \{n_1^2(t), n_2^2(t)\}}{\min_{0 \leq t \leq 1} r^5(t)}, \\ \left| \int_0^1 \frac{\partial U_i}{\partial n_j} dt' \right| &\leq \frac{3}{8} \frac{M^2 \max_{0 \leq t \leq 1} \{n_1^2(t), n_2^2(t)\}}{\min_{0 \leq t \leq 1} r^5(t)}, \end{aligned} \right\} \quad (22)$$

where M^2 denotes the greatest of the four following quantities:

$$2r_1^2 + r_2^2 + |(r_1 r_2)| ,$$

$$r_2^2 + |(r_1 r_2)| ,$$

$$r_1^2 + |(r_1 r_2)| ,$$

$$r_2^2 + 2r_2^2 + |(r_1 r_2)| .$$

We shall obtain the conditions for convergence of the process of successive approximations defined by formulas (20). We write them in the following form:

$$\left. \begin{aligned} n_1^n(t) &= n_1^0(t) + \tau^2 \int_0^1 U_1(t, t', n_1^{n-1}(t'), n_2^{n-1}(t')) dt' + \\ &\quad + \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_1}{\partial n_1} \right)_{n-1} \Delta_1^n + \left(\frac{\partial U_1}{\partial n_2} \right)_{n-1} \Delta_2^n \right\} dt' , \\ n_2^n(t) &= n_2^0(t) + \tau^2 \int_0^1 U_2(t, t', n_1^{n-1}(t'), n_2^{n-1}(t')) dt' + \\ &\quad + \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_2}{\partial n_1} \right)_{n-1} \Delta_1^n + \left(\frac{\partial U_2}{\partial n_2} \right)_{n-1} \Delta_2^n \right\} dt' . \end{aligned} \right\} \quad (23)$$

Subtracting $n_1^{n-1}(t)$ from $n_1^n(t)$ and $n_2^{n-1}(t)$ from $n_2^n(t)$, one obtains that

$$\begin{aligned} \Delta_1^n &= \tau^2 \int_0^1 [U_1(t, t', n_1^{n-1}(t'), n_2^{n-1}(t')) - U_1(t, t', n_1^{n-2}(t'), n_2^{n-2}(t'))] dt' - \\ &\quad - \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_1}{\partial n_1} \right)_{n-2} \Delta_1^{n-1} + \left(\frac{\partial U_1}{\partial n_2} \right)_{n-2} \Delta_2^{n-1} \right\} dt' + \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_1}{\partial n_1} \right)_{n-1} \Delta_1^n + \left(\frac{\partial U_1}{\partial n_2} \right)_{n-1} \Delta_2^n \right\} dt', \\ \Delta_2^n &= \tau^2 \int_0^1 [U_2(t, t', n_1^{n-1}(t'), n_2^{n-1}(t')) - U_2(t, t', n_1^{n-2}(t'), n_2^{n-2}(t'))] dt' - \\ &\quad - \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_2}{\partial n_1} \right)_{n-2} \Delta_1^{n-1} + \left(\frac{\partial U_2}{\partial n_2} \right)_{n-2} \Delta_2^{n-1} \right\} dt' + \tau^2 \int_0^1 \left\{ \left(\frac{\partial U_2}{\partial n_1} \right)_{n-1} \Delta_1^n + \left(\frac{\partial U_2}{\partial n_2} \right)_{n-1} \Delta_2^n \right\} dt'. \end{aligned}$$

Using the notation

$$\Delta^n = \max_{0 \leq t \leq 1} \{ |\Delta_1^n|, |\Delta_2^n| \},$$

$$\underline{n}^n = \max_{0 \leq t \leq 1} \{ n_1^n, n_2^n \},$$

and representing the difference

$$U_i(t, t', n_1^{n-1}(t'), n_2^{n-1}(t')) - U_i(t, t', n_1^{n-2}(t'), n_2^{n-2}(t'))$$

by Lagrange's formula in the form

$$\frac{\partial U_i}{\partial n_1} \Delta_1^{n-1} + \frac{\partial U_i}{\partial n_2} \Delta_2^{n-1},$$

where the derivatives $\frac{\partial U_i}{\partial n_j}$ ($j = 1, 2$) are taken at some intermediate points

$$n_i^n = n_i^{n-1} + \theta_{n_i} (n_i^{n-1} - n_i^{n-2}), \quad |\theta_{n_i}| \leq 1, \quad (i = 1, 2),$$

one has that

$$\Delta^n \leq \frac{\tau^2}{2} \frac{M^2 \max_{0 \leq t \leq 1} \{(\underline{n}^{n-2})^2, (\underline{n}^{n-1})^2\}}{\min_{0 \leq t \leq 1} r_{n-2}^5} \Delta^{n-1} +$$

$$+ \frac{\tau^2}{2} \frac{M^2 \max_{0 \leq t \leq 1} (\underline{n}^{n-2})^2}{\min_{0 \leq t \leq 1} r_{n-2}^5} \Delta^{n-1} + \frac{\tau^2}{2} \frac{M^2 \max_{0 \leq t \leq 1} (\underline{n}^{n-1})^2}{\min_{0 \leq t \leq 1} r_{n-1}^5} \Delta^n.$$

From this, it follows that

$$\Delta^n \leq \frac{\frac{\tau^2}{2} \frac{M^2 \max_{0 \leq t \leq 1} \{(\underline{n}^{n-2})^2, (\underline{n}^{n-1})^2\}}{\min_{0 \leq t \leq 1} \{r_{n-2}^5, r_{n-1}^5\}} + \frac{\tau^2}{2} \frac{M^2 \max_{0 \leq t \leq 1} (\underline{n}^{n-2})^2}{\min_{0 \leq t \leq 1} r_{n-2}^5} \Delta^{n-1}}{1 - \frac{\tau^2}{2} \frac{M^2 \max_{0 \leq t \leq 1} (\underline{n}^{n-1})^2}{\min_{0 \leq t \leq 1} r_{n-1}^5}} \leq$$

$$\leq \frac{\frac{\tau^2 M^2 \max_{0 \leq t \leq 1} \{(\underline{n}^{n-2})^2, (\underline{n}^{n-1})^2\}}{\min_{0 \leq t \leq 1} \{r_{n-2}^5, r_{n-1}^5\}} \Delta^{n-1}}{1 - \frac{\tau^2}{2} \frac{M^2 \max_{0 \leq t \leq 1} (\underline{n}^{n-1})^2}{\min_{0 \leq t \leq 1} r_{n-1}^5}} \leq \left(\frac{\frac{\tau^2 M^2 \max_{0 \leq t \leq 1} (\underline{n}^k)^2}{\min_{0 \leq t \leq 1} r_k^5}}{1 - \frac{\tau^2}{2} \frac{M^2 \max_{0 \leq t \leq 1} (\underline{n}^k)^2}{\min_{0 \leq t \leq 1} r_k^5}} \right)^{n-1} \Delta^1 \quad (24)$$

$$k = 0, 1, 2, \dots, n-1,$$

under the hypothesis that

$$\frac{\tau^2}{2} \frac{M^2 \max_{0 \leq t \leq 1} (\underline{n}^k)^2}{\min_{0 \leq t \leq 1} r_k^5} < 1. \quad (25)$$

We shall prove by mathematical induction that every

$$r_n \geq kh \quad \text{and} \quad \underline{n}^n \leq k' \underline{n}^0,$$

where k is a positive number less than 1, and k' is a positive number greater than 1.

For $n = 0$, these inequalities are obviously valid, in so far as $r_0 > h$.

We shall assume that it is valid for all r_j and \underline{n}^j for $j = 1, 2, \dots, n-1$, so that

$$r_j \geq kh, \quad \underline{n}^j \leq k' \underline{n}^0 \quad (j = 1, 2, \dots, n-1). \quad (26)$$

If it follows from this that the inequality (26) is satisfied for $j = n$, then it will be proved for all n .

By hypothesis,

$$\frac{\tau^2}{2} \frac{M^2 k'^2 n^0^2}{k^5 h^5} < 1 \quad (27)$$

by virtue of (24) and (26)

$$\Delta^n \leq \left(\frac{\frac{\tau^2 M^2 k'^2 n^0^2}{k^5 h^5}}{1 - \frac{\tau^2}{2} \frac{M^2 k'^2 n^0^2}{k^5 h^5}} \right)^{n-1} \Delta^1, \quad (28)$$

where, as can be seen from equation (16),

$$\Delta^1 \leq \frac{\frac{1}{8} \frac{\tau^2 \underline{n}^0}{r_0^3}}{1 - \frac{1}{2} \frac{\tau^2 M^2 \underline{n}^0^2}{r_0^5}}, \quad (29)$$

and r_0 and \underline{n}^0 satisfy the condition

$$\frac{\tau^2 M^2 \underline{n}^{0^2}}{2r_0^5} < 1.$$

Equation (23) and relationships (26), (28), (29) allow one to write the following inequality:

$$\begin{aligned} |\underline{n}^n - \underline{n}^0| \leq & \frac{1}{8} \frac{\tau^2 \max_{0 \leq t \leq 1} \underline{n}^{n-1}}{\min_{0 \leq t \leq 1} r^{n-1}} + \frac{1}{2} \frac{\tau^2 M^2 \max_{0 \leq t \leq 1} (\underline{n}^{n-1})^2}{\min_{0 \leq t \leq 1} r^{n-1}} \Delta^n \leq \frac{1}{8} \frac{\tau^2 k' \underline{n}^0}{k^3 h^3} + \\ & + \frac{1}{2} \frac{\tau^2 M^2 k' \underline{n}^{0^2}}{k^5 h^5} \cdot \left(\frac{\frac{\tau^2 M^2 k' \underline{n}^{0^2}}{k^5 h^5}}{1 - \frac{\tau^2 M^2 k' \underline{n}^{0^2}}{k^5 h^5}} \right)^{n-1} \cdot \frac{\frac{1}{8} \frac{\tau^2 k' \underline{n}^0}{k^3 h^3}}{1 - \frac{\tau^2 M^2 \underline{n}^{0^2}}{r_0^5}} \end{aligned}$$

It is required that

$$\frac{\frac{\tau^2 M^2 k' \underline{n}^{0^2}}{k^5 h^5}}{1 - \frac{\tau^2 M^2 k' \underline{n}^{0^2}}{k^5 h^5}} < 1,$$

or

$$\frac{3}{2} \frac{\tau^2 M^2 k' \underline{n}^{0^2}}{k^5 h^5} < 1. \quad (30)$$

Note that inequality (30) guarantees the satisfaction of (27). Now,

$$|\underline{n}^n - \underline{n}^0| \leq \frac{1}{8} \frac{\tau^2 k' \underline{n}^0}{k^3 h^3} + \frac{1}{3} \left[\frac{1}{1 - \frac{1}{2} \frac{\tau^2 M^2 \underline{n}^{0^2}}{r_0^5}} \right]$$

and

$$\frac{\underline{n}^n - \underline{n}^0}{k^3 h^3} + \frac{1}{8} \frac{\tau^2 k' \underline{n}^0}{k^3 h^3} \left[1 + \frac{1}{3} \frac{1}{1 - \frac{1}{2} \frac{\tau^2 M^2 \underline{n}^0{}^2}{r_0^5}} \right] .$$

We obtain an estimate for r_n ,

$$r_n = n_1^n r_1 + n_2^n r_2 = (n_1^n - n_1^0) r_1 + (n_2^n - n_2^0) r_2 + n_1^0 r_1 + n_2^0 r_2 =$$

$$= r_0 + (n_1^n - n_1^0) r_1 + (n_2^n - n_2^0) r_2 ,$$

$$|r_n - r_0| \leq |n^n - n^0| (r_1 + r_2) \leq \frac{1}{8} \frac{\tau^2 k' \underline{n}^0}{k^3 h^3} \left[1 + \frac{1}{3} \frac{1}{1 - \frac{1}{2} \frac{\tau^2 M^2 \underline{n}^0{}^2}{r_0^5}} \right] (r_1 + r_2)$$

and

$$r_n \geq r_0 - \frac{1}{8} \frac{\tau^2 k' \underline{n}^0}{k^3 h^3} \left[1 + \frac{1}{3} \frac{1}{1 - \frac{1}{2} \frac{\tau^2 M^2 \underline{n}^0{}^2}{r_0^5}} \right] (r_1 + r_2) .$$

In so far as $r_0 > h$,

$$r_n \geq h - \frac{1}{8} \frac{\tau^2 k' \underline{n}^0}{k^3 h^3} \left[1 + \frac{1}{3} \frac{1}{1 - \frac{1}{2} \frac{\tau^2 M^2 \underline{n}^0{}^2}{r_0^5}} \right] (r_1 + r_2) .$$

We determine k and k' so that

$$\frac{\underline{n}^0}{k^3 h^3} + \frac{1}{8} \frac{\tau^2 k' \underline{n}^0}{k^3 h^3} \left[1 + \frac{1}{3} \frac{1}{1 - \frac{1}{2} \frac{\tau^2 M^2 \underline{n}^0{}^2}{r_0^5}} \right] \leq k' \underline{n}^0 \quad (31)$$

Then $\underline{n} \leq k' \underline{n}^0$, $r_n \geq kh$, which proves inequality (26) for arbitrary n .

Inequalities (31) and (32) may be written in the following way:

$$k' \geq \frac{1}{1 - \frac{1}{8} \frac{\tau^2}{k^3 h^3}} \left[1 + \frac{1}{3} \frac{1}{1 - \frac{1}{2} \frac{\tau^2 M^2 \underline{n}^0{}^2}{r_0^5}} \right], \quad (33)$$

$$k \leq 1 - \frac{1}{4} \frac{\tau^2 k' \underline{n}^0}{k^3 h^3} \left[1 + \frac{1}{3} \frac{1}{1 - \frac{1}{2} \frac{\tau^2 M^2 \underline{n}^0{}^2}{r_0^5}} \right]. \quad (34)$$

If k and k' may be chosen in order that (30), (33), (34) are satisfied, then the process of successive approximations defined by formulas (20) converges.

From (33) and (34), assuming that equality holds, k and k' may be determined by the method of successive approximations, taking $k = k' = 1$ as initial approximations.

It is possible to obtain convergence conditions more simply, which, however, are cruder.

If one chooses initially that

$$\frac{1}{2} \frac{\tau^2 M^2 \underline{n}^0{}^2}{r_0^5} < \frac{1}{2},$$

then for this, the satisfaction of inequalities (33), (34) is sufficient to guarantee the satisfaction of the following inequalities:

$$k' \geq \frac{1}{1 - \frac{5}{24} \frac{\tau^2}{k^3 h^3}},$$

$$k \leq \frac{1}{1 - \frac{5}{12} \frac{\tau^2 k'^0}{k^3 h^3}}.$$

Setting $k = \frac{1}{2}$, $k' = \frac{3}{2}$, the convergence conditions take the form

$$h^3 > 10\tau^2,$$

$$h^5 > 108M^2\tau^2.$$

By virtue of inequality (24), if condition (30) is satisfied, the sequences of functions $\{n_1^n\}$, $\{n_2^n\}$ converge to functions n_1, n_2 , which are solutions of the system (18).

One may convince oneself of this immediately by letting $n \rightarrow \infty$ in equations (20).

The legality of passing to the limit under the integral sign follows from the uniform convergence of the functions in the integrand, in as much as the estimate (24) is independent of time.

We shall prove the uniqueness of the solution obtained in the region

$$\frac{3}{2} \frac{\tau^2 M^2 \max_{0 \leq t \leq 1} n^2}{\min_{0 \leq t \leq 1} r^5} < 1.$$

Suppose that there are other solutions n_1', n_2' which satisfy condition (30).

Consider the differences

$$n_1' - n_1 = \tau^2 \int_0^1 [U_1(t, t', n_1'(t'), n_2'(t')) - U_1(t, t', n_1(t'), n_2(t'))] dt',$$

$$n_2' - n_2 = \tau^2 \int_0^1 [U_2(t, t', n_1'(t'), n_2'(t')) - U_2(t, t', n_1(t'), n_2(t'))] dt'.$$

Replace the increment of the functions $U_1(t, t', n_1'(t'), n_2'(t')) - U_1(t, t', n_1(t'), n_2(t'))$ by the differential at the point $n_i'' = n_i + \theta_{n_i}(n_i' - n_i)$, $|\theta_{n_i}| \leq 1$, ($i = 1, 2$) and use the inequality (22) for

$$|\underline{n}' - \underline{n}| = \max_{0 \leq t \leq 1} \{ |n_1' - n_1|, |n_2' - n_2| \},$$

which may be written

$$|\underline{n}' - \underline{n}| \leq \frac{1}{2} \frac{\tau^2 M^2 \max_{0 \leq t \leq 1} \underline{n}^2}{\min_{0 \leq t \leq 1} r^5} |\underline{n}' - \underline{n}|,$$

or

$$|\underline{n}' - \underline{n}| \left(1 - \frac{1}{2} \frac{\tau^2 M^2 \max_{0 \leq t \leq 1} \underline{n}^2}{\min_{0 \leq t \leq 1} r^5} \right) \leq 0.$$

In as much as the expression in parentheses is positive, one obtains

$$|\underline{n}' - \underline{n}| \leq 0,$$

which, by the virtue of the non-negativity of $|\underline{n}' - \underline{n}|$ is only possible if

$$|\underline{n}' - \underline{n}| \equiv 0.$$

In order to obtain an estimate for the error committed if the calculation is stopped at the n th approximation, we consider the difference

$$|\underline{n} - \underline{n}^n| = \max \{ |n_1 - n_1^n|, |n_2 - n_2^n| \} \leq \sum_{k=n}^{\infty} \Delta^k = \frac{\alpha^{n-1}}{1-\alpha} \Delta^1,$$

where

$$\alpha = \frac{\frac{\tau^2 M^2 k'^2 \underline{n}_0^2}{k^5 h^5}}{1 - \frac{\tau^2 M^2 k'^2 \underline{n}_0^2}{k^5 h^5}},$$

k and k' satisfy inequalities (33), (34), h satisfies (30), and Δ^1 satisfies inequality (29).

§5. The Case of Perturbed Motion.

If in addition to the attraction of the central body, a force $m\tilde{F}$ acts on the planet or satellite, then, instead of equation (1), the following equation of motion holds:

$$\ddot{\underline{r}}(t) = k^2(1+m) \frac{\underline{r}(t)}{r^3(t)} + \tilde{F}(t). \quad (35)$$

The corresponding integral equation may be written in the form

$$\underline{r}(t) = \underline{r}_0(t) + \tau^2 \int_0^1 K(t, t') \left\{ \frac{\underline{r}(t')}{r^3(t')} - \tilde{F}(t') \right\} dt'. \quad (36)$$

If the force $m\tilde{F}$ is the perturbing action of a planet with mass m_s and radius vector \underline{r}_s , equation (36) takes the form

$$\underline{r}(t) = \underline{r}_0(t) + \tau^2 \int_0^1 K(t, t') \left\{ \frac{\underline{r}(t')}{|\underline{r}(t')|^3} + \frac{m_s}{1+m} \left[\frac{\underline{r}(t') - \underline{r}_s(t')}{|\underline{r}(t') - \underline{r}_s(t')|^3} + \frac{\underline{r}_s(t')}{r_s^3(t')} \right] \right\} dt'. \quad (37)$$

Set

$$\underline{r}(t) = n_1(t) \underline{r}_1 + n_2(t) \underline{r}_2 + n_s(t) \underline{r}_s,$$

then equation (37) reduces to a system of three equations for the quantities

$n_1(t)$, $n_2(t)$, $n_s(t)$:

$$n_1(t) = n_1^0(t) + \tau^2 \int_0^1 K(t, t') \frac{n_1(t')}{r^3(t')} \left[1 + \frac{m_s}{1+m} \frac{r^3(t')}{|\underline{r}(t') - \underline{r}_s(t')|^3} \right] dt',$$

$$n_2(t) = n_2^0(t) + \tau^2 \int_0^1 K(t, t') \frac{n_2(t')}{r^3(t')} \left[1 + \frac{m_s}{1+m} \frac{r^3(t')}{|\underline{r}(t') - \underline{r}_s(t')|^3} \right] dt',$$

$$n_s(t) = \tau^2 \int_0^1 K(t, t') \left\{ \frac{n_s(t')}{r^3(t')} + \frac{m_s}{1+m} \frac{n_s(t') - 1}{|\underline{r}(t') - \underline{r}_s(t')|^3} + \frac{m_s}{1+m} \frac{1}{r_s^3(t')} \right\} dt',$$

which may be solved by Newton's method.

As initial approximations one may take $n_1^0, n_2^0, n_s = 0$ as in the case of unperturbed motion.

§6. Determination of Orbits from Three Observations

Let t_1, t_2, t_3 ($t_1 \leq t_2 \leq t_3$) denote the times of three observations chosen for the determination of an orbit. Each of the three observations permits writing a relationship

$$\underline{r}_i = \rho_i \underline{e}_i - \underline{R}_i \quad (i = 1, 2, 3), \quad (38)$$

where $\rho_i \underline{e}_i$ denotes the geocentric radius vector of the planet (here, \underline{e}_i is the unit vector indicating the direction to the planet, ρ_i is the geocentric distance),

and \underline{R}_1 is the geocentric solar radius vector.

For the determination of the orbit given the directions \underline{e}_1 to the planet and the geocentric solar radius vectors \underline{R}_i ($i = 1, 2, 3$), the boundary values \underline{r}_1 and \underline{r}_3 may themselves be unknowns, and have to be determined from some system of equations.

To obtain these equations, one uses the relationships which connect the geocentric coordinates for three given times

$$\underline{r}_2 = n_1(t_2)\underline{r}_1 + n_3(t_2)\underline{r}_3$$

or, by virtue of (38),

$$\rho_2 \underline{e}_2 - \underline{R}_2 = n_1(t_2)(\rho_1 \underline{e}_1 - \underline{R}_1) + n_3(t_2)(\rho_2 \underline{e}_2 - \underline{R}_2).$$

Multiplying this equation first by $\frac{[\underline{e}_2 \underline{e}_3]}{\underline{e}_1[\underline{e}_2 \underline{e}_3]}$, then by $\frac{[\underline{e}_1 \underline{e}_2]}{\underline{e}_1[\underline{e}_2 \underline{e}_3]}$, one obtains

$$\left. \begin{aligned} n_1(t_2)\rho_1 &= [n_1(t_2)\underline{R}_1 + n_3(t_2)\underline{R}_3 - \underline{R}_2] \frac{[\underline{e}_2 \underline{e}_3]}{\underline{e}_1[\underline{e}_2 \underline{e}_3]}, \\ n_3(t_2)\rho_2 &= [n_1(t_2)\underline{R}_1 + n_3(t_2)\underline{R}_3 - \underline{R}_2] \frac{[\underline{e}_1 \underline{e}_2]}{\underline{e}_1[\underline{e}_2 \underline{e}_3]}, \end{aligned} \right\} \quad (39)$$

$$\underline{e}_1[\underline{e}_2 \underline{e}_3] \neq 0.$$

The solution of system (39), together with the system

$$\left. \begin{aligned} n_1(t) &= n_1^0(t) + \tau^2 \int_0^1 K(t, t') \frac{n_1(t')}{|n_1(t')\underline{r}_1 + n_2(t')\underline{r}_2|^3} dt', \\ n_3(t) &= n_3^0(t) + \tau^2 \int_0^1 K(t, t') \frac{n_3(t')}{|n_1(t')\underline{r}_1 + n_3(t')\underline{r}_3|^3} dt', \end{aligned} \right\} \quad (40)$$

determines ρ_1 and ρ_3 , and, consequently, \underline{r}_1 and \underline{r}_3 .

As the initial approximation for the ratio of the area of the triangles, one chooses n_1^0, n_3^0 , which are ratios of intermediate times

$$n_1^0 = \frac{\tau_1}{\tau_2}, \quad n_3^0 = \frac{\tau_3}{\tau_2},$$

where

$$\tau_1 = k\sqrt{1+m} (t_3 - t_2),$$

$$\tau_2 = k\sqrt{1+m} (t_3 - t_1),$$

$$\tau_3 = k\sqrt{1+m} (t_2 - t_1).$$

Substituting them into equation (39), one obtains ρ_1, ρ_3 in the first approximation.

Approximate values of $\underline{r}_1, \underline{r}_3$ are determined with the aid of relationships (38), and equations (40) are solved with them to obtain new values of n_1, n_3 , and so on. Therefore, equations (38)-(40) together permit the determination of the geocentric distances ρ_1, ρ_2, ρ_3 , and, consequently, the geocentric radius vectors $\underline{r}_1, \underline{r}_2, \underline{r}_3$ of the planet. Knowing these values, one may determine the elements of the orbit in the ordinary way.

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